

Embeddings of spaces of quregisters into special linear groups

Dalia Cervantes and Guillermo Morales-Luna

Computer Science Department, CINVESTAV-IPN, Mexico City, Mexico
dalia@computacion.cs.cinvestav.mx, gmorales@cs.cinvestav.mx

November 15, 2016

Abstract

We study embeddings of the unit sphere of complex Hilbert spaces of dimension a power 2^n into the corresponding groups of non-singular linear transformations. For the case of $n = 1$, the sphere $S_1(\mathbb{C})$ of qubits is identified with $SU(2)$ and the algebraic structure of this last group is carried into $S_1(\mathbb{C})$. Hence it is natural to analyse whether is it possible, for $n \geq 2$, to carry the structure of the symmetry group $SU(2^n)$ into the unit sphere $S_{2^n-1}(\mathbb{C})$. For $n = 2$ the embeddings of $S_3(\mathbb{C})$ into $GL(2^2)$, obtained as tensor products of the above embedding, fails to determine a bijection between $S_3(\mathbb{C})$ and $SU(2^2)$, but they determine entanglement measures consistent with von Neumann entropy.

1 Introduction

The basic information particle in Quantum Computing is the so called *qubit* which can be realised as a normalised linear combination of an “up” ($|0\rangle$) and “down” ($|1\rangle$) state for, let us say, the spin of an electron. Formally, the unit sphere $S_1(\mathbb{C})$ of the complex Hilbert space $\mathbb{H}_1 = \mathbb{C}^2$ consists of the *qubits*.

The space \mathbb{H}_1 has dimension 2 and its canonical basis is $\{|0\rangle, |1\rangle\}$, where

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mathbf{e}_0, \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathbf{e}_1.$$

Any basis $\{\mathbf{x}_0, \mathbf{x}_1\}$ of \mathbb{H}_1 is said *positively oriented* if the change of basis matrix $[\mathbf{x}_0 \ \mathbf{x}_1]$, with respect to the canonical basis, has determinant equal to 1. Any qubit $\mathbf{x}_0 \in S_1(\mathbb{C})$ can be associated to a second qubit $\mathbf{x}_1 \in S_1(\mathbb{C})$ such that $\{\mathbf{x}_0, \mathbf{x}_1\}$ is positively oriented.

The unit circle in the complex plane \mathbb{C} has a natural group structure with complex multiplication. By associating to each qubit the positively oriented basis consisting of itself and its orthogonal complement (in the positive sense), then a natural identification of $S_1(\mathbb{C})$ with $SU(2)$ results. Hence, $S_1(\mathbb{C})$ inherits the algebraic structure of the symmetries group $SU(2)$. This suggests the possibility to carry Quantum Computing into the group $SU(2)$. However this is not possible with the proposed identification because a computer gate $U \in U(2)$ commutes with the embedding $S_1(\mathbb{C}) \rightarrow SU(2)$ if and only if U preserves orientation, namely, $\det U = 1$.

The composition of qubits produces more complex information structures.

Let \mathbb{H}_n denote the n -fold tensor power of \mathbb{H}_1 . The elements of its unit sphere $S_{2^n-1}(\mathbb{C}) \subset \mathbb{H}_n$ will be called *n-quregisters*. The n -fold tensor product $(S_1(\mathbb{C}))^{\otimes n}$ of $S_1(\mathbb{C})$ is included in $S_{2^n-1}(\mathbb{C})$. Hence, the n -fold tensor power of qubits are *n-quregisters*. Let us call an *n-quregister separable* if it is the n -fold tensor powers of qubits, i. e. unit vectors in \mathbb{H}_1 .

The main motivation of the current research is to provide the unit sphere of the 2^n -dimensional complex Hilbert space of an algebraic structure as a homomorphic image of the group structure of the symmetry group $SU(2^n)$.

Naturally, an embedding of the sphere $S_{2^n-1}(\mathbb{C})$ of *n-quregisters* into the space of non-singular linear transforms $SL(2^n)$ is sought to be congruent with tensor products. This motivates the introduction of the maps in relations (9)-(12) below, for $n = 2$. However, the images of the maps meet $SL(2^2) - U(2^2)$, and they are useless to transport the algebraic structure of either $SU(2^2)$ or $U(2^2)$ into $S_3(\mathbb{C})$.

Nevertheless, it is possible to introduce a measure of entanglement in $S_3(\mathbb{C})$ through these maps.

Several criteria have been introduced for entanglement measurement [2, 3, 5]. The introduced entanglement measure in this paper is consistent with the notion of partial separability through von Neumann entropy of partial traces.

2 Preliminaries

Let us recall some basic notions.

Let \mathbb{U} be a finite-dimensional complex Hilbert space, and $m = \dim(\mathbb{U})$, then $\mathbb{U} \approx \mathbb{C}^m$. Let $\mathcal{L}(\mathbb{U}) = \{T : \mathbb{U} \rightarrow \mathbb{U} \mid T \text{ is linear}\}$ be the space of linear maps defined on \mathbb{U} . Provided with the *Hilbert-Schmidt inner product* $(T, S) \mapsto \langle T | S \rangle = \text{Tr} T^H S$, it is a complex Hilbert space of dimension m^2 . By fixing a basis in \mathbb{U} , there is a natural identification $\mathcal{L}(\mathbb{U}) \approx \mathbb{C}^{m \times m}$: the space of square $(m \times m)$ -complex matrices.

An operator $T \in \mathcal{L}(\mathbb{U})$ is *positive (semidefinite)* if $\exists S \in \mathcal{L}(\mathbb{U})$: $T = S^H S$, and it is *positive definite* if, besides, it is non-singular. Let $\text{Pos}(\mathbb{U})$ be the collection of positive operators.

A *quantum state* is a positive operator $T \in \text{Pos}(\mathbb{U})$ such that $\text{Tr} T = 1$. Let $\text{D}(\mathbb{U})$ be the collection of quantum states.

Now, let $S_{\mathbb{U}} = \{\mathbf{x} \in \mathbb{U} \mid \mathbf{x}^H \mathbf{x} = 1\}$ be the unit sphere in \mathbb{U} . Clearly, $S_{\mathbb{U}} \approx S_{m-1}(\mathbb{C})$: the unit sphere in the m -dimensional complex linear space.

The map $\rho : S_{\mathbb{U}} \rightarrow \text{D}(\mathbb{U})$, $\mathbf{x} \mapsto \rho(\mathbf{x}) = \mathbf{x} \mathbf{x}^H$, is an embedding (namely, an injective map). For each $\mathbf{x} \in S_{\mathbb{U}}$, $\rho(\mathbf{x}) \in \mathcal{L}(\mathbb{U})$ is the orthogonal projection along the ray spanned by \mathbf{x} in the space \mathbb{U} . Seen as a matrix, $\rho(\mathbf{x})$ is called the *density matrix* determined by the unit vector $\mathbf{x} \in S_{\mathbb{U}}$.

Then any unit vector $\mathbf{x} \in S_{\mathbb{U}}$ can be considered as a state.

A state $T \in \text{D}(\mathbb{U})$ is *pure* if $T \in \text{Image}(\rho)$, namely, $\exists \mathbf{x} \in S_{\mathbb{U}}$: $\rho(\mathbf{x}) = T$. Non-pure states are called *mixed states*, as well.

Let \mathbb{V} be another finite-dimensional complex Hilbert space, with $n = \dim(\mathbb{V})$. Then the tensor product $\mathbb{U} \otimes \mathbb{V}$ has dimension mn .

An unit vector $\mathbf{u} \in S_{\mathbb{U} \otimes \mathbb{V}}$ is *separable* if $\exists (\mathbf{x}, \mathbf{y}) \in S_{\mathbb{U}} \times S_{\mathbb{V}}$: $\mathbf{u} = \mathbf{x} \otimes \mathbf{y}$.

Proposition 2.1 *If $(\mathbf{x}, \mathbf{y}) \in \mathbb{U} \times \mathbb{V}$ then $\rho(\mathbf{x} \otimes \mathbf{y}) = \rho(\mathbf{x}) \otimes \rho(\mathbf{y})$.*

An operator $T \in \mathcal{L}(\mathbb{U} \otimes \mathbb{V})$ is *separable* if there are two sequences $U_0, \dots, U_{k-1} \in \mathcal{L}(\mathbb{U})$, $V_0, \dots, V_{k-1} \in \mathcal{L}(\mathbb{V})$ such that $T = \sum_{\kappa=0}^{k-1} U_{\kappa} \otimes V_{\kappa}$.

Proposition 2.2 *For any $\mathbf{u} \in S_{\mathbb{U} \otimes \mathbb{V}}$, if \mathbf{u} is separable (as a unit vector) then $\rho(\mathbf{u})$ is separable in $\text{D}(\mathbb{U} \otimes \mathbb{V})$.*

Let us consider now the two-dimensional complex Hilbert space $\mathbb{H}_1 = \mathbb{C}^2$ and its tensor powers, $\forall n > 1$: $\mathbb{H}_n = \mathbb{H}_{n-1} \otimes \mathbb{H}_1$. Clearly, $\dim \mathbb{H}_n = 2^n$. The unit sphere $S_{2^n-1}(\mathbb{C})$ of \mathbb{H}_n is the set of n -quregisters.

Let $Q = \{0, 1\}$ be the set of classical bits and let $\{|0\rangle, |1\rangle\}$ denote the canonical basis of \mathbb{H}_1 .

For any $n \geq 1$ and any $\varepsilon = \varepsilon_{n-1} \dots \varepsilon_1 \varepsilon_0 \in Q^n$ let $|\varepsilon\rangle = |\varepsilon_{n-1}\rangle \otimes \dots \otimes |\varepsilon_1\rangle \otimes |\varepsilon_0\rangle$. Then $(|\varepsilon\rangle)_{\varepsilon \in Q^n}$ is the canonical basis of \mathbb{H}_n .

For any $n \geq 2$ and any $\varepsilon = \varepsilon_{n-1} \dots \varepsilon_1 \varepsilon_0 \in Q^n$ let $\mathbf{b}_{\varepsilon} = \frac{1}{\sqrt{2}} (|0\varepsilon_{n-2} \dots \varepsilon_1 \varepsilon_0\rangle + (-1)^{\varepsilon_{n-1}} |1\overline{\varepsilon_{n-2}} \dots \overline{\varepsilon_1} \overline{\varepsilon_0}\rangle)$, where $\overline{\varepsilon_i}$ is the orthogonal complement element of ε_i , for each $i = n-2, \dots, 0$.

Then $(\mathbf{b}_{\varepsilon})_{\varepsilon \in Q^n}$ is the Bell basis of \mathbb{H}_n , and it consists of maximally entangled states.

The group $\text{U}(2^n)$ consists of all unitary linear transforms $\mathbb{H}_n \rightarrow \mathbb{H}_n$, with map composition as operation, and $\text{SU}(2^n)$ is the subgroup of orientation preserving unitary transforms: $\forall T \in \text{SU}(2^n)$, $\det T = 1$.

Let $S_1(\mathbb{C})$ the unit sphere of $\mathbb{H}_1 = \mathbb{C}^2$, $S_1(\mathbb{C}) = \{\mathbf{x} \in \mathbb{H}_1 \mid \mathbf{x}^H \mathbf{x} = 1\}$ is the set of *qubits*. The tensor square power $\mathbb{H}_2 = \mathbb{H}_1 \otimes \mathbb{H}_1$ is the four-dimensional complex Hilbert space. Its unit sphere $S_3(\mathbb{C}) = \{\mathbf{x} \in \mathbb{H}_2 \mid \mathbf{x}^H \mathbf{x} = 1\}$ is the set of *2-quregisters*. In this case, $\mathcal{L}(\mathbb{H}_2) \approx \mathbb{C}^{2^2 \times 2^2}$ is the space of square $(2^2 \times 2^2)$ -complex matrices and $\text{D}(\mathbb{H}_2)$ is the set of positive matrices with trace 1. For any 2-quregister $\mathbf{x} = (x_0, x_1, x_2, x_3) \in S_3(\mathbb{C})$ we have

$$\rho(\mathbf{x}) = \mathbf{x} \mathbf{x}^H = \begin{bmatrix} |x_0|^2 & x_0 \overline{x_1} & x_0 \overline{x_2} & x_0 \overline{x_3} \\ x_1 \overline{x_0} & |x_1|^2 & x_1 \overline{x_2} & x_1 \overline{x_3} \\ x_2 \overline{x_0} & x_2 \overline{x_1} & |x_2|^2 & x_2 \overline{x_3} \\ x_3 \overline{x_0} & x_3 \overline{x_1} & x_3 \overline{x_2} & |x_3|^2 \end{bmatrix}.$$

The reduced trace matrices are

$$\text{Tr}_0(\rho(\mathbf{x})) = \begin{bmatrix} |x_0|^2 + |x_1|^2 & x_0\bar{x}_2 + x_1\bar{x}_3 \\ x_2\bar{x}_0 + x_3\bar{x}_1 & |x_2|^2 + |x_3|^2 \end{bmatrix}, \quad \text{Tr}_1(\rho(\mathbf{x})) = \begin{bmatrix} |x_0|^2 + |x_2|^2 & x_0\bar{x}_1 + x_2\bar{x}_3 \\ x_1\bar{x}_0 + x_3\bar{x}_2 & |x_1|^2 + |x_3|^2 \end{bmatrix}.$$

The reduced trace matrices have the same eigenvalues. Those are

$$\lambda_0 = \frac{1}{2}(1 - s(\mathbf{x})) \quad , \quad \lambda_1 = \frac{1}{2}(1 + s(\mathbf{x})) \quad (1)$$

where

$$s(\mathbf{x}) = \sqrt{1 - 4|t(\mathbf{x})|^2} \quad \text{and} \quad t(\mathbf{x}) = x_0x_3 - x_1x_2, \quad (2)$$

Hence, the Von Neumann entropy of the reduced trace matrices is

$$\begin{aligned} E(\rho(\mathbf{x})) &= -\lambda_0 \log_2 \lambda_0 - \lambda_1 \log_2 \lambda_1 \\ &= -\frac{1}{2}(1 - s(\mathbf{x}))(\log_2(1 - s(\mathbf{x})) - 1) - \frac{1}{2}(1 + s(\mathbf{x}))(\log_2(1 + s(\mathbf{x})) - 1) \\ &= -\frac{1}{2} \left(\log_2((1 - s(\mathbf{x}))(1 + s(\mathbf{x}))) + s(\mathbf{x}) \log_2 \left(\frac{1 + s(\mathbf{x})}{1 - s(\mathbf{x})} \right) \right) + 1 \\ &= -\frac{1}{2} \left(\log_2(1 - s(\mathbf{x})^2) + s(\mathbf{x}) \log_2 \left(\frac{1 - s(\mathbf{x})^2}{(1 - s(\mathbf{x}))^2} \right) \right) + 1 \\ &= -\frac{1}{2}(1 + s(\mathbf{x})) \log_2(1 - s(\mathbf{x})^2) + s(\mathbf{x}) \log_2(1 - s(\mathbf{x})) + 1 \end{aligned}$$

which is a measure of the entanglement of $\rho(\mathbf{x})$ in $D(\mathbb{H}_2)$. Detailed definitions and constructions of the above results are shown in Section 4.2.

On the other hand, from (1) we see that $\lambda_0 + \lambda_1 = 1$, namely $\lambda_1 = 1 - \lambda_0$ and

$$\begin{aligned} E(\rho(\mathbf{x})) &= -\lambda_0 \log_2 \lambda_0 - (1 - \lambda_0) \log_2(1 - \lambda_0) \\ &= -\frac{1}{2}((1 - s(\mathbf{x})) \log_2(1 - s(\mathbf{x})) + (1 + s(\mathbf{x})) \log_2(1 + s(\mathbf{x}))) + 1. \end{aligned} \quad (3)$$

Thus,

$$E(\rho(\mathbf{x})) = 0 \iff \lambda_0 \in \{0, 1\} \iff s(\mathbf{x}) = \sqrt{1} \iff t(\mathbf{x}) = 0 \iff x_1x_2 = x_0x_3, \quad (4)$$

and this last condition entails that $\mathbf{x} \in S_3(\mathbb{C})$ is a separable unit vector. These separability criteria are included in the Proposition 4.1.

3 Case of qubits

3.1 Embedding $S_1(\mathbb{C})$ into $SU(2)$

Let us define

$$\Psi_1 : S_1(\mathbb{C}) \rightarrow SU(2) \quad , \quad \mathbf{x} = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} \mapsto \Psi_1(\mathbf{x}) = \begin{bmatrix} x_0 & -\bar{x}_1 \\ x_1 & \bar{x}_0 \end{bmatrix}. \quad (5)$$

Via the map Ψ_1 , any qubit is identified with an element of $SU(2)$. Conversely, if $X = \begin{bmatrix} x_0 & y_0 \\ x_1 & y_1 \end{bmatrix} \in SU(2)$, then $x_0y_1 - x_1y_0 = 1$. By assuming $(x_0, x_1) \in S_1(\mathbb{C})$, the solutions of this last equation are the points $(y_0, y_1) \in \mathbb{H}_1$ in the straight line passing through $(-\bar{x}_1, \bar{x}_0)$ parallel to the straight-line orthogonal to $(x_0, -x_1)$. Since $(x_0, -x_1) \in S_1(\mathbb{C})$, this line is tangent to $S_1(\mathbb{C})$ at this point. The solution line is parameterised thus as $Y(y) = (-\bar{x}_1, \bar{x}_0) + y(1, -x_0^{-1}x_1)$, with $y \in \mathbb{C}$. Also $(-\bar{x}_1, \bar{x}_0) \in S_1(\mathbb{C})$, and the only solution (y_0, y_1) of $x_0y_1 - x_1y_0 = 1$ in $S_1(\mathbb{C})$ is $(y_0, y_1) = (-\bar{x}_1, \bar{x}_0)$. Thus $X = \Psi_1(x_0, x_1)$. Hence, Ψ_1 is a bijection $S_1(\mathbb{C}) \rightarrow SU(2)$.

The operation in the group $SU(2)$ translated into $S_1(\mathbb{C})$ is

$$\star_1 : S_1(\mathbb{C}) \times S_1(\mathbb{C}) \rightarrow S_1(\mathbb{C}) \quad , \quad \left(\begin{bmatrix} x_{00} \\ x_{10} \end{bmatrix}, \begin{bmatrix} x_{01} \\ x_{11} \end{bmatrix} \right) \mapsto \begin{bmatrix} x_{00} \\ x_{10} \end{bmatrix} \star_1 \begin{bmatrix} x_{01} \\ x_{11} \end{bmatrix} = \begin{bmatrix} x_{00}x_{01} - \bar{x}_{10}x_{11} \\ x_{10}x_{01} + \bar{x}_{00}x_{11} \end{bmatrix}, \quad (6)$$

hence $(S_1(\mathbb{C}), \star_1, \begin{bmatrix} 1 \\ 0 \end{bmatrix})$ is a group.

In fact, for a qubit $\mathbf{x} = [x_0 \ x_1]^T \in S_1(\mathbb{C})$ we have

$$\begin{aligned} \mathbf{x}^{\star_1 1} &= \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}, \\ \mathbf{x}^{\star_1 2} &= \begin{bmatrix} x_0^2 - x_1 \overline{x_1} \\ 2x_1 \Re(x_0) \end{bmatrix}, \\ \mathbf{x}^{\star_1 3} &= \begin{bmatrix} x_0^3 - x_1(2\Re(x_0) + x_0)\overline{x_1} \\ x_1 \left((\overline{x_0})^2 + 2x_0 \Re(x_0) - x_1 \overline{x_1} \right) \end{bmatrix}, \\ \mathbf{x}^{\star_1 4} &= \begin{bmatrix} (x_0^2 - x_1 \overline{x_1})^2 - 4x_1 \Re(x_0)^2 \overline{x_1} \\ x_1 \left(x_0^3 + x_0 |x_0|^2 + 2\Re(x_0) \left((\overline{x_0})^2 - 2x_1 \overline{x_1} \right) \right) \end{bmatrix}, \end{aligned}$$

where $\Re(z)$ denotes the real part of the complex number $z \in \mathbb{C}$.

We recall that the *order* of an element $\mathbf{x} \in S_1(\mathbb{C})$ is $o(\mathbf{x}) = \min_{n \in \mathbb{N}} \{ \mathbf{x}^{\star_1 n} = [1 \ 0]^T \}$. For instance,

$$o\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = 1, \quad o\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) = 2, \quad o\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = 4, \quad o\left(\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) = 4,$$

while

$$\forall \varepsilon_0, \varepsilon_1 \in \{-1, +1\} : o\left(\frac{1}{\sqrt{2}} \begin{bmatrix} \varepsilon_0 \\ \varepsilon_1 \end{bmatrix}\right) = 8.$$

As a direct consequence of the *Poincaré's Recurrence Theorem* [4] we have:

Proposition 3.1 *Let $r_0, r_1 \in [0, 1]$ be such that $|r_0|^2 + |r_1|^2 = 1$ and $t_0, t_1 \in \mathbb{R}$ two irrational numbers. Let $\mathbf{x} = [r_0 \exp(i t_0) \ r_1 \exp(i t_1)]^T$, where $i = \sqrt{-1}$. Then the subgroup $\langle \mathbf{x} \rangle = \{\mathbf{x}^{\star_1 n} \mid n \in \mathbb{Z}\} < S_1(\mathbb{C})$, generated by \mathbf{x} , is a countable dense subgroup of $S_1(\mathbb{C})$.*

We recall that a *quantum gate* U is a unitary map $U : \mathbb{H}_1 \rightarrow \mathbb{H}_1$, namely $U^H U = \mathbb{I}_2$. Thus $U \in U(2)$ and it is a bijection $S_1(\mathbb{C}) \rightarrow S_1(\mathbb{C})$ when restricted to the unit sphere $S_1(\mathbb{C})$.

A mechanical computation suffices to prove the following:

Proposition 3.2 *A quantum gate U commutes with the bijection $\Psi_1 : S_1(\mathbb{C}) \rightarrow SU(2)$ if and only if $\det U = 1$. In symbols:*

$$\forall U \in U(2) : [\Psi_1 \circ U = U \circ \Psi_1 \iff U \in SU(2)],$$

where \circ is the composition of maps.

4 Case of 2-quregisters

4.1 Embedding $S_3(\mathbb{C})$ into $SL(2^2)$

Let $\mathbf{c}_0 = [c_{00} \ c_{10}]^T$, $\mathbf{c}_1 = [c_{01} \ c_{11}]^T$ be two qubits in the unit sphere $S_1(\mathbb{C})$ of the Hilbert space \mathbb{H}_1 and let $\Psi_1 : S_1(\mathbb{C}) \rightarrow SU(2)$ be the bijection defined as in (5). Then,

$$\Psi_1(\mathbf{c}_0) \otimes \Psi_1(\mathbf{c}_1) = \begin{bmatrix} c_{00}c_{01} & -c_{00}\overline{c_{11}} & -\overline{c_{10}}c_{01} & \overline{c_{10}}c_{11} \\ c_{00}c_{11} & c_{00}\overline{c_{01}} & -\overline{c_{10}}c_{11} & -\overline{c_{10}}c_{01} \\ c_{10}c_{01} & -c_{10}\overline{c_{11}} & \overline{c_{00}}c_{01} & -\overline{c_{00}}c_{11} \\ c_{10}c_{11} & c_{10}\overline{c_{01}} & \overline{c_{00}}c_{11} & \overline{c_{00}}c_{01} \end{bmatrix}. \quad (7)$$

The collection of 2-quregisters is the unit sphere $S_3(\mathbb{C})$ of the Hilbert space \mathbb{H}_2 . A natural embedding of the Cartesian product $S_1(\mathbb{C}) \times S_1(\mathbb{C})$ into $S_3(\mathbb{C})$ is given by the injective map

$$I_2 : S_1(\mathbb{C}) \times S_1(\mathbb{C}) \rightarrow S_3(\mathbb{C}) \quad , \quad (\mathbf{c}_0, \mathbf{c}_1) \mapsto I_2(\mathbf{c}_0, \mathbf{c}_1) = \mathbf{c}_0 \otimes \mathbf{c}_1. \quad (8)$$

A 2-quregister $\mathbf{x} \in S_3(\mathbb{C})$ is *separable* if it is in the image of I_2 , namely, there exist $\mathbf{c}_0, \mathbf{c}_1 \in S_1(\mathbb{C})$ such that $\mathbf{x} = \mathbf{c}_0 \otimes \mathbf{c}_1$.

Proposition 4.1 A 2-quregister $\mathbf{x} = [x_0 \ x_1 \ x_2 \ x_3]^T \in S_3(\mathbb{C})$ is separable if and only if $x_0x_3 = x_1x_2$.

Let $Sp_2 \subset S_3(\mathbb{C})$ be the collection of 2-quregisters that are separable. Thus, the condition at Proposition 4.1 is a defining predicate of the set Sp_2 .

For each index $j \in \{0, 1, 2, 3\}$, let $C_j = \{\mathbf{x} \in S_3(\mathbb{C}) \mid x_j \neq 0\} = S_3(\mathbb{C}) \cap \pi_j(\mathbb{C} - \{0\})$, where π_j is the j -th canonical projection. Each set C_j is an open set in the unit sphere $S_3(\mathbb{C})$, with the topology induced by the Hilbert space \mathbb{H}_2 , and they cover $S_3(\mathbb{C})$. Namely, $(C_j)_{j=0}^3$ is an open covering of $S_3(\mathbb{C})$ and it determines a structure of a complex 3-dimensional differential manifold in $S_3(\mathbb{C})$: each set C_j is a 3-dimensional complex chart in $S_3(\mathbb{C})$.

Proposition 4.2 If a 2-quregister $\mathbf{x} = [x_0 \ x_1 \ x_2 \ x_3]^T \in S_3(\mathbb{C})$ is separable, then it can be tensor splitted as $\mathbf{x} = \mathbf{c}_0 \otimes \mathbf{c}_1$, where the qubits $\mathbf{c}_0, \mathbf{c}_1$ are determined according to the following rules:

$$\begin{aligned} \mathbf{x} \in C_0 &\implies \mathbf{c}_0 = \frac{1}{r_{02}} \begin{bmatrix} x_0 \\ x_2 \end{bmatrix} \wedge \mathbf{c}_1 = r_{02} \begin{bmatrix} 1 \\ \frac{x_1}{x_0} \end{bmatrix} \quad \text{with } r_{02} = \sqrt{|x_0|^2 + |x_2|^2}, \\ \mathbf{x} \in C_1 &\implies \mathbf{c}_0 = \frac{1}{r_{13}} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} \wedge \mathbf{c}_1 = r_{13} \begin{bmatrix} \frac{x_0}{x_1} \\ 1 \end{bmatrix} \quad \text{with } r_{13} = \sqrt{|x_1|^2 + |x_3|^2}, \\ \mathbf{x} \in C_2 &\implies \mathbf{c}_0 = \frac{1}{r_{02}} \begin{bmatrix} x_0 \\ x_2 \end{bmatrix} \wedge \mathbf{c}_1 = r_{02} \begin{bmatrix} 1 \\ \frac{x_3}{x_2} \end{bmatrix}, \\ \mathbf{x} \in C_3 &\implies \mathbf{c}_0 = \frac{1}{r_{13}} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} \wedge \mathbf{c}_1 = r_{13} \begin{bmatrix} \frac{x_2}{x_3} \\ 1 \end{bmatrix}. \end{aligned}$$

Besides since $\mathbf{x} = \mathbf{c}_0 \otimes \mathbf{c}_1$, for any unit complex number $u \in \mathbb{C}$, $\mathbf{x} = (u^{-1}\mathbf{c}_0) \otimes (u\mathbf{c}_1)$ is another tensor split of \mathbf{x} .

For $k \in \{0, 1, 2, 3\}$ and a unit complex number $u \in \mathbb{C}$, let $\Phi_{2ku} : C_k \rightarrow \mathbb{C}^{2^2 \times 2^2}$ be defined as follows, with $\mathbf{x} \in C_k$:

$$\Phi_{20u}(\mathbf{x}) = \begin{bmatrix} x_0 & -u^{-2}\xi(x_0)^2\overline{x_1} & -u^2\overline{x_2} & \overline{x_3} \\ x_1 & u^{-2}x_0 & -u^2\xi(x_2)^{-2}x_3 & -\overline{x_2} \\ x_2 & -u^{-2}\xi\left(\frac{x_1}{x_0}\right)^{-2}x_3 & u^2\overline{x_0} & -\overline{x_1} \\ x_3 & u^{-2}x_2 & u^2\xi(x_0)^{-2}x_1 & \overline{x_0} \end{bmatrix}, \quad (9)$$

$$\Phi_{21u}(\mathbf{x}) = \begin{bmatrix} x_0 & -u^{-2}x_1 & -u^2\xi(x_3)^{-2}x_2 & \overline{x_3} \\ x_1 & u^{-2}\xi(x_1)^2\overline{x_0} & -u^2\overline{x_3} & -\overline{x_2} \\ x_2 & -u^{-2}x_3 & u^2\xi(x_1)^{-2}x_0 & -\overline{x_1} \\ x_3 & u^{-2}\xi\left(\frac{x_0}{x_1}\right)^{-2}x_2 & u^2\overline{x_1} & \overline{x_0} \end{bmatrix}, \quad (10)$$

$$\Phi_{22u}(\mathbf{x}) = \begin{bmatrix} x_0 & -u^{-2}\xi\left(\frac{x_3}{x_2}\right)^{-2}x_1 & -u^2\overline{x_2} & \overline{x_3} \\ x_1 & u^{-2}x_0 & -u^2\xi(x_2)^{-2}x_3 & -\overline{x_2} \\ x_2 & -u^{-2}\xi(x_2)^2\overline{x_3} & u^2\overline{x_0} & -\overline{x_1} \\ x_3 & u^{-2}x_2 & u^2\xi(x_0)^{-2}x_1 & \overline{x_0} \end{bmatrix}, \quad (11)$$

$$\Phi_{23u}(\mathbf{x}) = \begin{bmatrix} x_0 & -u^{-2}x_1 & -u^2\xi(x_3)^{-2}x_2 & \overline{x_3} \\ x_1 & u^{-2}\xi\left(\frac{x_2}{x_3}\right)^{-2}x_0 & -u^2\overline{x_3} & -\overline{x_2} \\ x_2 & -u^{-2}x_3 & u^2\xi(x_1)^{-2}x_0 & -\overline{x_1} \\ x_3 & u^{-2}\xi(x_3)^2\overline{x_2} & u^2\overline{x_1} & \overline{x_0} \end{bmatrix}, \quad (12)$$

where $\xi : \mathbb{C} \rightarrow \mathbb{C}$ is the map that for any non-zero complex number takes the unit complex number along its own direction,

$$z \mapsto \xi(z) = \begin{cases} \frac{z}{|z|} & \text{if } z \neq 0, \\ 1 & \text{if } z = 0. \end{cases}$$

Direct computations show that for any separable 2-quregister $\mathbf{x} \in Sp_2$, since the condition at Proposition 4.1 holds, for any $k \in \{0, 1, 2, 3\}$, $\Phi_{2ku}(\mathbf{x})^H \Phi_{2ku}(\mathbf{x}) = \mathbb{I}_{2^2}$. Consequently for $k \in \{0, 1, 2, 3\}$, the map $\Phi_{2ku} : C_k \rightarrow \mathbb{C}^{2^2 \times 2^2}$ is such that it determines an embedding of $C_k \cap Sp_2$ into the symmetry group $SU(2^2)$.

In the case of a separable 2-quregister $\mathbf{x} \in C_k \cap Sp_2$, for a tensor split $\mathbf{x} = \mathbf{c}_0 \otimes \mathbf{c}_1$ we have that the matrix $\Psi_1(\mathbf{c}_0) \otimes \Psi_1(\mathbf{c}_1)$ at (7) coincides with $\Phi_{23u}(\mathbf{x})$ being $u = \xi(c_k)$ the unit complex number in the direction of the complex number $c_k \in \mathbb{C}$, where $c_0 = c_{00}$, $c_1 = c_{10}$, $c_2 = c_{01}$, and $c_3 = c_{11}$.

Example 4.1.1. The i -th vector $\mathbf{e}_i = [\delta_{ij}]_{j=0}^3$ in the canonical basis of \mathbb{H}_2 is a separable 2-quregister:

$$\mathbf{e}_{2i_1+i_0} = \mathbf{e}_{i_1} \otimes \mathbf{e}_{i_0}.$$

We have $\mathbf{e}_i \in C_i$ while $\mathbf{e}_i \notin C_j$, for $j \neq i$. Then for any unit complex number $u \in \mathbb{C}$:

$$\begin{aligned} \Phi_{20u}(\mathbf{e}_0) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & u^{-2} & 0 & 0 \\ 0 & 0 & u^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, & \Phi_{21u}(\mathbf{e}_1) &= \begin{bmatrix} 0 & -u^{-2} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & u^2 & 0 \end{bmatrix}, \\ \Phi_{22u}(\mathbf{e}_2) &= \begin{bmatrix} 0 & 0 & -u^2 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & u^{-2} & 0 & 0 \end{bmatrix}, & \Phi_{23u}(\mathbf{e}_3) &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -u^2 & 0 \\ 0 & -u^{-2} & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \end{aligned} \tag{13}$$

Being separable the vectors at the canonical basis, all matrices above are unitary.

Example 4.1.2. The i -th vector \mathbf{b}_i in the Bell basis of \mathbb{H}_2 is a maximally entangled 2-quregister:

$$\mathbf{b}_{2i_1+i_0} = \frac{1}{\sqrt{2}} (\mathbf{e}_0 \otimes \mathbf{e}_{i_1} + (-1)^{i_0} \mathbf{e}_1 \otimes \mathbf{e}_{1+i_1}).$$

Each 2-quregister \mathbf{b}_i is in two charts C_k . For any unit complex number $u \in \mathbb{C}$:

$$\begin{aligned} \Phi_{20u}(\mathbf{b}_0) = \Phi_{23u}(\mathbf{b}_0) &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & u^{-2} & -u^2 & 0 \\ 0 & -u^{-2} & u^2 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, & \Phi_{20u}(\mathbf{b}_1) = \Phi_{23u}(\mathbf{b}_1) &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & u^{-2} & u^2 & 0 \\ 0 & u^{-2} & u^2 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}, \\ \Phi_{21u}(\mathbf{b}_2) = \Phi_{22u}(\mathbf{b}_2) &= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -u^{-2} & -u^2 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & u^{-2} & u^2 & 0 \end{bmatrix}, & \Phi_{21u}(\mathbf{b}_3) = \Phi_{22u}(\mathbf{b}_3) &= \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -u^{-2} & u^2 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 \\ 0 & -u^{-2} & u^2 & 0 \end{bmatrix}. \end{aligned}$$

No above matrix is unitary. In fact, if B is any of the above matrices, then $B^H B$ has 2 and 0 as eigenvalues, each of multiplicity 2. Hence, the spectral norm of B is $\|B\|_2 = \sqrt{2}$.

Example 4.1.3. Consider a vector $\mathbf{x}_p = (\sqrt{p}, 0, 0, \sqrt{1-p}) = \sqrt{p} \mathbf{e}_0 + \sqrt{1-p} \mathbf{e}_3 \in S_3(\mathbb{C})$, with $p \in [0, 1]$. Then $\mathbf{x}_p \in C_0 \cap C_3$. For any unit complex number $u \in \mathbb{C}$ we have

$$\Phi_{20u}(\mathbf{x}_p) = \Phi_{23u}(\mathbf{x}_p) = \begin{bmatrix} \sqrt{p} & 0 & 0 & \sqrt{1-p} \\ 0 & u^{-2}\sqrt{p} & -u^2\sqrt{1-p} & 0 \\ 0 & -u^{-2}\sqrt{1-p} & u^2\sqrt{p} & 0 \\ \sqrt{1-p} & 0 & 0 & \sqrt{p} \end{bmatrix}.$$

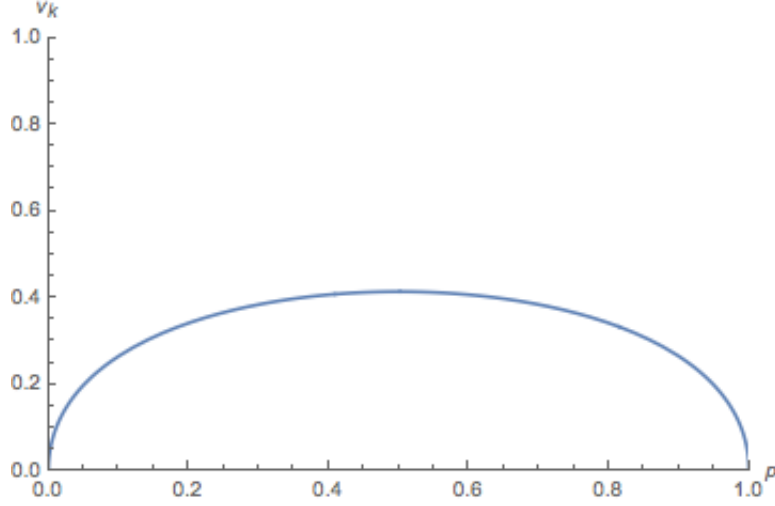


Figure 1: Graph of the spectral norm of the transforms of the 2-quregister \mathbf{x}_p translated by -1 .

In fact, from relations (13), we have

$$\Phi_{20u}(\mathbf{x}_p) = \sqrt{p} \Phi_{20u}(\mathbf{e}_0) + \sqrt{1-p} \Phi_{23u}(\mathbf{e}_3).$$

Then, for the matrix

$$V = \begin{bmatrix} 1 & 0 & 0 & 2\sqrt{1-p}\sqrt{p} \\ 0 & 1 & -2\sqrt{1-p}\sqrt{p} & 0 \\ 0 & -2\sqrt{1-p}\sqrt{p} & 1 & 0 \\ 2\sqrt{1-p}\sqrt{p} & 0 & 0 & 1 \end{bmatrix},$$

we have $\Phi_{20u}(\mathbf{x}_p)^H \Phi_{20u}(\mathbf{x}_p) = V = \Phi_{23u}(\mathbf{x}_p)^H \Phi_{23u}(\mathbf{x}_p)$, and the eigenvalues of V are $1 - 2\sqrt{(1-p)p}$, $1 + 2\sqrt{(1-p)p}$, each with characteristic 2. Hence, the spectral norm is $\|\Phi_{20u}(\mathbf{x}_p)\|_2 = \sqrt{1 + 2\sqrt{(1-p)p}}$.

We observe that for $p = 0$, $\mathbf{x}_0 = \mathbf{e}_3$, the fourth vector in the canonical basis, which is separable, for $p = \frac{1}{2}$, $\mathbf{x}_{\frac{1}{2}} = \mathbf{b}_0$, the first vector in the Bell basis, which is maximally entangled and for $p = 1$, $\mathbf{x}_1 = \mathbf{e}_0$, the first vector in the canonical basis, which is separable.

In Figure 1 it is displayed the plot of the map $p \mapsto \|\Phi_{20u}(\mathbf{x}_p)\|_2 - 1 = \sqrt{1 + 2\sqrt{(1-p)p}} - 1$.

For any 2-quregister $\mathbf{x} \in C_k$, let $\nu_k(\mathbf{x}) = \|\Phi_{2ku}(\mathbf{x}_p)\|_2 - 1$, with $u \in \mathbb{C}$ being a unit complex number. As a characterisation of the set $Sp_2 \subset S_3(\mathbb{C})$ of separable 2-quregisters, we have:

Proposition 4.3 *For any $k \in \{0, 1, 2, 3\}$, any unitary complex number $u \in \mathbb{C}$ and any $\mathbf{x} \in C_k$:*

$$\mathbf{x} \in Sp_2 \iff \Phi_{2ku}(\mathbf{x}) \in SU(2^2) \iff \nu_k(\mathbf{x}) = 0.$$

The map ν_k can be considered a measure of entanglement. It satisfies the conventional conditions of an entanglement measure [1]:

Separability. If $\mathbf{x} \in S_3(\mathbb{C})$ is separable, then $\nu_k(\mathbf{x}) = 0$.

Normality. In the maximally entangled vectors, ν_k attains its maxima. Indeed, the measure $\frac{2}{\sqrt{2}-1}\nu_k$ has $2 = \log_2 2^2$ as maximal value.

Continuity. ν_k is continuous with respect to the topology of $S_3(\mathbb{C})$.

Boundedness under local operations. The entanglement cannot be increased by applying local operations.

Let us check this last assertion.

For each $k \in \{0, 1, 2, 3\}$ and a unit complex number $u \in \mathbb{C}$, let $\Phi_{2ku} : C_k \rightarrow \mathbb{C}^{2^2 \times 2^2}$ be defined by relations (9, 10, 11, 12) respectively.

First let us state as a proposition the following result, which can be proved in an exhaustive way through direct calculations.

Proposition 4.4 *For each $k \in \{0, 1, 2, 3\}$ and each $\mathbf{x} \in C_k$ there is a unit complex number $u = u(\mathbf{x}) \in \mathbb{C}$, such that*

$$\Phi_{2ku}^H(\mathbf{x}) \Phi_{2ku}(\mathbf{x}) \begin{bmatrix} 1 & 0 & 0 & 2\overline{t(\mathbf{x})} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2t(\mathbf{x}) & 0 & 0 & 1 \end{bmatrix} = Z_u(\mathbf{x})$$

where $t(\mathbf{x})$ is defined by (2),

Proposition 4.5 *The spectrum of the matrix $Z_u(\mathbf{x})$ is $\Lambda_u(\mathbf{x}) = \{1, 1, 1 - 2|t(\mathbf{x})|, 1 + 2|t(\mathbf{x})|\}$, and the corresponding eigenvectors are the columns of the matrix*

$$W_u(\mathbf{x}) = \begin{bmatrix} 0 & 0 & -\frac{\overline{t(\mathbf{x})}}{|t(\mathbf{x})|} & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{t(\mathbf{x})}{|t(\mathbf{x})|} \end{bmatrix}.$$

Hence the introduced measure ν_k is such that

$$\nu_k(\mathbf{x}) = \sqrt{1 + 2|t(\mathbf{x})|} - 1. \quad (14)$$

Let us compare roughly $\nu_k(\mathbf{x})$ as in (14) with the Von Neumann entropy $E(\rho(\mathbf{x}))$ as in (3). According to Proposition 2.2 and the equivalences in (4):

$$\mathbf{x} \text{ is separable} \iff E(\rho(\mathbf{x})) = 0 \iff t(\mathbf{x}) = 0 \iff \nu_k(\mathbf{x}).$$

On the other hand,

- according to (14), maximal values of $\nu_k(\mathbf{x})$ correspond to maximal values of $t(\mathbf{x})$,
- according to (2), if $t(\mathbf{x}) = \frac{1}{2}$, then $s(\mathbf{x}) = 0$, $\lambda_0 = \frac{1}{2}$ and both $E(\rho(\mathbf{x})) = 1$ and $\nu_k(\mathbf{x}) = \sqrt{2} - 1$ attain their maximum values, and
- according to (2), if $|t(\mathbf{x})| > \frac{1}{2}$, then $s(\mathbf{x}), \lambda_0 \in \mathbb{C} - \mathbb{R}$, thus $E(\rho(\mathbf{x}))$ and $\nu_k(\mathbf{x})$ cannot further be compared.

Proposition 4.6 *For any linear $U \in \mathcal{L}(\mathbb{H}_1)$ and any unitary vector $\mathbf{x} \in S_3(\mathbb{C})$,*

$$t((U \otimes \mathbb{I}_2)\mathbf{x}) = (\det U)t(\mathbf{x}) = t((\mathbb{I}_2 \otimes U)\mathbf{x}).$$

Hence, if U is unitary, then

$$|t((U \otimes \mathbb{I}_2)\mathbf{x})| = |t(\mathbf{x})| = |t((\mathbb{I}_2 \otimes U)\mathbf{x})|.$$

Proposition 4.7 *The introduced measure, ν_k is not increasing under the application of local operators with classical communication (LOCC), or stochastic LOCC (SLOCC).*

Finally, since the spectral bound of a matrix $M = (m_{ij})_{i,j \in \{0,1,2,3\}} \in \mathbb{C}^{2^2 \times 2^2}$ is bounded as

$$\|M\|_2 \leq \left(\sum_{i,j=0}^3 |m_{ij}|^2 \right)^{\frac{1}{2}},$$

from relations (9)–(12) we have

$$\forall k \in \{0, 1, 2, 3\}, u \in \mathbb{C} \text{ with } |u| = 1, \mathbf{x} \in C_k : \|\Phi_{2ku}(\mathbf{x})\|_2 \leq 2,$$

but this is not a tight bound.

4.2 Density matrices

Let $\mathbf{x} = (x_0, x_1, x_2, x_3) \in S_3(\mathbb{C})$ be a 2-quregister. The *projection* along the direction of this vector is $\pi_{\mathbf{x}} : \mathbb{H}_2 \rightarrow \mathbb{H}_2$, $\mathbf{y} \mapsto \pi_{\mathbf{x}}(\mathbf{y}) = (\mathbf{x} \mathbf{x}^H) \mathbf{y}$. The matrix $\rho_2(\mathbf{x}) = \mathbf{x} \mathbf{x}^H$ is the *density matrix* determined by \mathbf{x} . The notations ρ_2 and ρ_1 refer to given in the Preliminaries (2), where the subindices emphasize the domains \mathbb{H}_2 and \mathbb{H}_1 respectively of the projections.

Similarly for a qubit $\mathbf{z} \in S_1(\mathbb{C})$, its density matrix is $\rho_1(\mathbf{z}) = \mathbf{z} \mathbf{z}^H$.

A mechanical computation suffices to prove the following:

Proposition 4.8 *If a separable 2-quregister is factored as in Proposition 4.2, say $\mathbf{x} = \mathbf{c}_0 \otimes \mathbf{c}_1$, then*

$$\rho_2(\mathbf{x}) = \rho_1(\mathbf{c}_0) \otimes \rho_1(\mathbf{c}_1).$$

A *mixed 2-quregister* is a convex combination of density matrices. Let $\mathbf{m} = \sum_{i \in I} p_i \rho_2(\mathbf{x}_i) \in \mathbb{C}^{2^2 \times 2^2}$ be a mixed state, $\forall i \in I$, $p_i \in [0, 1]$ and $\sum_{i \in I} p_i = 1$. Naturally, if, in an extreme case, for some index i_0 we have $p_{i_0} = 1$ and $p_i = 0$, for all $i \neq i_0$, then \mathbf{m} is a *pure state*. A well known characterisation of pure states is the following:

$$\mathbf{m} \in \mathbb{C}^{2^2 \times 2^2} \text{ is pure} \iff \mathbf{m}^2 = \mathbf{m} \iff \text{Tr}(\mathbf{m}^2) = 1.$$

The mixed 2-quregister \mathbf{m} is *separable* if $\forall i \in I$, $\mathbf{x}_i = \mathbf{c}_{i0} \otimes \mathbf{c}_{i1}$, with $\mathbf{c}_{i0}, \mathbf{c}_{i1} \in S_1(\mathbb{C})$, or equivalently $\rho_2(\mathbf{x}_i) = \rho_1(\mathbf{c}_{i0}) \otimes \rho_1(\mathbf{c}_{i1})$.

Any separable mixed state is actually a *density operator*: it is symmetric, positive and with trace 1. The mixed state is determined by a $(2^2 \times 2^2)$ -complex matrix and such matrix determines as well a sesquilinear map $\mathbb{H}_1 \times \mathbb{H}_1 \rightarrow \mathbb{C}$, $(\mathbf{x}, \mathbf{y}) \mapsto B(\mathbf{x}, \mathbf{y}) = \mathbf{x}^H Q \mathbf{y}$, which in turn determines the quadratic form $\mathbf{x} \mapsto Q(\mathbf{x}) = \mathbf{x}^H Q \mathbf{x}$. The *partial traces*, regarded as quadratic forms, are

$$\begin{aligned} \text{Tr}_0 Q : \mathbb{H}_1 &\rightarrow \mathbb{C} \quad , \quad \mathbf{z}_0 \mapsto \text{Tr}_0 Q(\mathbf{z}_0) = \sum_{k=0}^1 (\mathbf{z}_0 \otimes \mathbf{e}_k)^H Q(\mathbf{z}_0 \otimes \mathbf{e}_k), \\ \text{Tr}_1 Q : \mathbb{H}_1 &\rightarrow \mathbb{C} \quad , \quad \mathbf{z}_1 \mapsto \text{Tr}_1 Q(\mathbf{z}_1) = \sum_{k=0}^1 (\mathbf{e}_k \otimes \mathbf{z}_1)^H Q(\mathbf{e}_k \otimes \mathbf{z}_1). \end{aligned}$$

(the indexes at the argument variables \mathbf{z} , are referring to the corresponding subsystems of the composed system in \mathbb{H}_2). Thus, regarded as matrices, their corresponding entries are

$$\forall i, j \in \{0, 1\} : (\text{Tr}_0 Q)_{ij} = q_{2i, 2j} + q_{2i+1, 2j+1} \quad \& \quad (\text{Tr}_1 Q)_{ij} = q_{i, j} + q_{2+i, 2+j}.$$

The *von Neumann entropy* of the whole system is

$$E(\mathbf{m}) = - \sum_{j=0}^3 \lambda_j \log_2 \lambda_j,$$

where $(\lambda_j)_{j=0}^3$ is the collection of eigenvalues of \mathbf{m} and, similarly, the *reduced von Neumann entropy* of each reduced subsystem $\text{Tr}_i \mathbf{m}$ is

$$E_i(\mathbf{m}) = - \sum_{j=0}^1 \lambda_{ij} \log_2 \lambda_{ij},$$

where $(\lambda_{ij})_{j=0}^1$ is the collection of eigenvalues of $\text{Tr}_i \mathbf{m}$, $i = 0, 1$. The reduced von Neumann entropies entail a measure of entanglement, and, according to the Uniqueness Theorem [1], sufficient and necessary conditions determine whether any other entanglement measure coincide with this criterion.

Example. Consider the vector $\mathbf{x}_p = (\sqrt{p}, 0, 0, \sqrt{1-p}) = \sqrt{p} \mathbf{e}_0 + \sqrt{1-p} \mathbf{e}_3 \in S_3(\mathbb{C})$, with $p \in [0, 1]$, as in Example 4.1.3. Then,

$$\mathbf{m}_p = \rho_2(\mathbf{x}_p) = \begin{bmatrix} p & 0 & 0 & \sqrt{1-p}\sqrt{p} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{1-p}\sqrt{p} & 0 & 0 & 1-p \end{bmatrix},$$

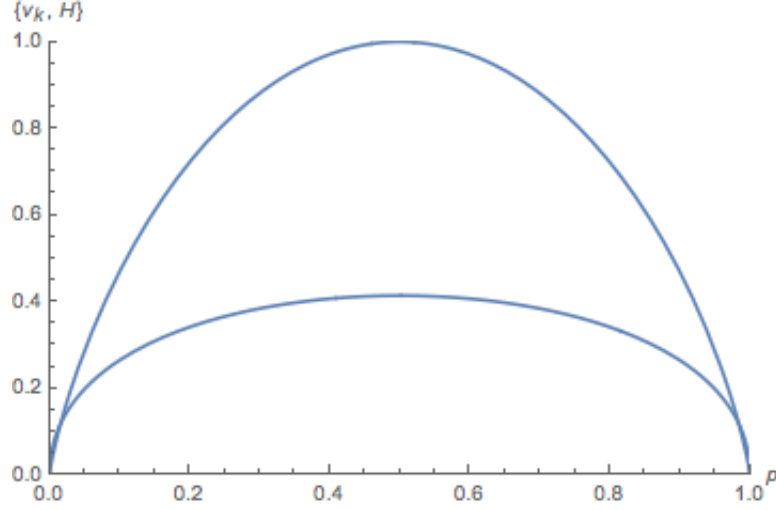


Figure 2: Graphs of H and ν_0 . The maxima are $H(\frac{1}{2}) = 1$ and $\nu_0(\frac{1}{2}) = \sqrt{2} - 1$.

which is an idempotent matrix, $\mathbf{m}_p^2 = \mathbf{m}_p$, hence it is a pure state (it can be identified with \mathbf{x}_p), it has eigenvalues 0 and 1 of respective multiplicities 3 and 1, and eigenspaces $\mathcal{L}(\mathbf{y}_p, \mathbf{e}_1, \mathbf{e}_2)$ and $\mathcal{L}(\mathbf{x}_p)$ where $\mathbf{y}_p = (\sqrt{1-p}, 0, 0, -\sqrt{p})$ is orthogonal to \mathbf{x}_p . The von Neumann entropy of \mathbf{m}_p is thus 0. Now, the first partial trace is

$$(\text{Tr}_0 \mathbf{m}_p) = \begin{bmatrix} p & 0 \\ 0 & 1-p \end{bmatrix},$$

hence $E(\text{Tr}_0 \mathbf{m}_p) = -p \log_2 p - (1-p) \log_2 (1-p) = H(p)$, where H is Shannon's entropy function. $E(\text{Tr}_0 \mathbf{m}_p)$ is indeed a measure of the entanglement of the 2-quregister \mathbf{x}_p and it is consistent with the measure $\nu_k(\mathbf{x}_p)$ as shown in Example 4.1.3 (see Figure 2). With this criterion, the vectors at the Bell basis correspond to maximally entangled states.

5 Conclusion

We have analysed an embedding of the unit sphere of the 2^2 -dimensional complex Hilbert space into the symmetry group $\text{SU}(2^2)$.

It is rather usual to present geometrically the collection of qubits, namely, the unit sphere $S_1(\mathbb{C})$ of \mathbb{H}_1 , as the Bloch sphere, but little attention is paid to the possible algebraic structures within $S_1(\mathbb{C})$.

For the case $n = 1$ there is a natural identification Ψ_1 of the sphere S_2 with $\text{SU}(2)$, although the algebraic structure of $\text{SU}(2)$ is not consistent with the application of quantum gates in $S_1(\mathbb{C})$, in fact only the unitary operators that preserve orientation commute with the identification Ψ_1 .

Unfortunately for $n \geq 2$, the natural embedding Ψ_n obtained by the tensor product of the former bijection Ψ_1 may fail to define a bijection between $S_{2^n-1}(\mathbb{C})$ and $\text{SU}(2^n)$. In this paper, we have shown that actually for $n = 2$, the embedding Ψ_2 does not determine a bijection between $S_3(\mathbb{C})$ and $\text{SU}(2^2)$.

However, the proposed operators satisfy the desired embedding when they are restricted to the separable n -quregisters. These operators give rise to entanglement measures which are compatible with conventional entanglement measures, as von Neumann entropy.

The procedures used in this paper are rather standard and most probably can be generalised to the quregisters of any length. We look towards to formally prove this sketch of research.

References

- [1] Matthew J. Donald, Michal Horodecki, and Oliver Rudolph. The uniqueness theorem for entanglement measures. *Journal of Mathematical Physics*, 43(9), 2002.
- [2] Wiesław Laskowski, Marcin Markiewicz, Danny Rosseau, Tim Byrnes, Kamil Kostrzewa, and Adrian Kołodziejcki. Correlation-based entanglement criterion in bipartite multiboson systems. *Phys. Rev. A*, 92:022339, Aug 2015.
- [3] Michael Seevinck and Jos Uffink. Partial separability and entanglement criteria for multiqubit quantum states. *Phys. Rev. A*, 78:032101, Sep 2008.
- [4] Elias M. Stein and Rami Shakarchi. *Real analysis: measure theory, integration, and Hilbert spaces*. Princeton Lectures in Analysis. Princeton University Press, Princeton (N.J.), Oxford, 2005.
- [5] Sabine Wölk, Marcus Huber, and Otfried Gühne. Unified approach to entanglement criteria using the cauchy-schwarz and hölder inequalities. *Phys. Rev. A*, 90:022315, Aug 2014.